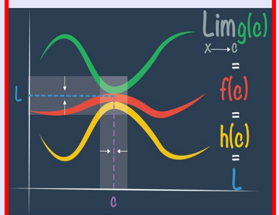
Calculus I

Lecture 7



Feb 19-8:47 AM

Open notes class Quiz 7

Prove $\lim\limits_{x \to -2} (5x - 3) = -13.$

$f(x) = 5x - 3$ ✓ $\quad L = -13$ ✓✓ $\quad a = -2$ ✓

$\lim\limits_{x \to -2} (5x-3) = 5(-2) - 3 = \boxed{-13}$ ✓✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|5x - 3 - (-13)| < \varepsilon$ 〃 $|x - (-2)| < \delta$

$|5x - 3 + 13| < \varepsilon$ 〃 $|x + 2| < \delta$

$|5x + 10| < \varepsilon$ 〃 $|x+2| < \delta$

$|5(x+2)| < \varepsilon$

$5|x+2| < \varepsilon \rightarrow |x+2| < \frac{\varepsilon}{5}$

Pick $\delta = \frac{\varepsilon}{5}$

Mar 3-11:06 AM

Prove $\lim\limits_{x\to 3}(x^2+10x+25)=64$

1) $f(x)=x^2+10x+25$

$L=64$

$a=3$

2) $\lim\limits_{x\to3}(x^2+10x+25)=3^2+10(3)+25$

$=9+30+25$

$=9+55$

$=64$ ✓

3) $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|x^2+10x+25-64|<\varepsilon$ 〃 $|x-3|<\delta$

$|x^2+10x-39|<\varepsilon$ 〃 $|x-3|<\delta$

$|(x+13)(x-3)|<\varepsilon$ 〃 $|x-3|<\delta$

$|x+13|\,|x-3|<\varepsilon$ 〃 $|x-3|<\delta$

Bound Keep

$\boxed{\text{If } |x|<K,\ K>0, \text{ then } -K<x<K}$

with polynomial function $\Rightarrow$ If $\delta\le 1$

$|x-3|<1$

Now

$|x+13|\,|x-3|<17|x-3|<\varepsilon$

$-1<x-3<1$

Add 3

$2<x<4$

Add 13

$15<x+13<17$

$|x+13|<17$

Divide by 17

$|x-3|<\dfrac{\varepsilon}{17}$

$\boxed{\text{Pick } \delta=\min\left\{1,\dfrac{\varepsilon}{17}\right\}}$

Mar 5-8:59 AM

Prove $\lim\limits_{x\to\frac12}\dfrac{1}{2x}=1$

$f(x)=\dfrac{1}{2x}$ $L=1$ ✓ $a=\dfrac12$

$\lim\limits_{x\to\frac12}\dfrac{1}{2x}=\dfrac{1}{2(\frac12)}=\dfrac11=1$ ✓

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$\left|\dfrac{1}{2x}-1\right|<\varepsilon$ 〃 $\left|x-\dfrac12\right|<\delta$

$\left|\dfrac{1}{2x}-\dfrac{2x}{2x}\right|<\varepsilon$

$\left|\dfrac{1-2x}{2x}\right|<\varepsilon$

$|a-b|=|b-a|$

$\left|\dfrac{2x-1}{2x}\right|<\varepsilon$

$\left|\dfrac{\cancel{2}(x-\frac12)}{\cancel{2}x}\right|<\varepsilon$

$\left|\dfrac{a}{b}\right|=\dfrac{|a|}{|b|}$

$\dfrac{|x-\frac12|}{|x|}<\varepsilon$

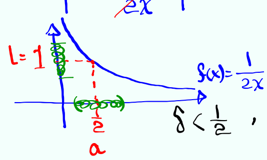
$\dfrac{1}{|x|}\cdot\left|x-\dfrac12\right|<\varepsilon$

If $\dfrac{1}{|x|}<C$, then

$\dfrac{1}{|x|}\cdot\left|x-\dfrac12\right|<C\left|x-\dfrac12\right|<\varepsilon$

$\left|x-\dfrac12\right|<\dfrac{\varepsilon}{C}$

Bound



$\delta<\dfrac12$, Pick $\delta\le\dfrac14$

Mar 5-9:11 AM

If $\delta \le \frac{1}{4}$

then $|x - \frac{1}{2}| < \frac{1}{4}$ $\quad -\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4}$

$-.25 < x - .5 < .25$

Add $.5$

$\frac{1}{4} \rightarrow .25 < x < .75 \leftarrow \frac{3}{4}$

we had

$\frac{1}{|x|} \cdot |x - \frac{1}{2}| < 4|x - \frac{1}{2}| < \varepsilon$ $\quad 4 > \frac{1}{x} > \frac{4}{3}$

$|x - \frac{1}{2}| < \frac{\varepsilon}{4}$ $\quad \frac{4}{3} < \frac{1}{x} < 4$

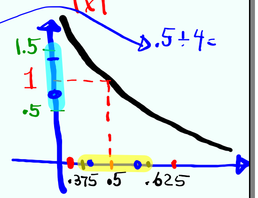
Pick $\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{4}\right\}$ $\quad \frac{1}{|x|} < 4$

$.5 \div 4 =$

$\varepsilon = .5$ $\quad \delta = \min\left\{\frac{1}{4}, \frac{.5}{4}\right\}$

$= \min\left\{\frac{1}{4}, \frac{1}{8}\right\}$

$= \frac{1}{8} = .125$



$1.5 = \frac{1}{2x}$ | $.5 = \frac{1}{2x}$ | $x = .38$ | $x = .6$

$1.5(2x) = 1$ | $.5(2x) = 1$ | $\frac{1}{2(.38)} = 1.3$ | $\frac{1}{2(.6)} = .83$

$3x = 1$ | $x = 1$

$x = \frac{1}{3} = .\bar{3}$

Mar 5-9:22 AM

Prove $\lim_{x \to 0} \sqrt[3]{x} = 0$

$f(x) = \sqrt[3]{x}$ $\quad L = 0 ✓$ $\quad a = 0$

$\lim_{x \to 0} \sqrt[3]{x} = \sqrt[3]{0} = 0 ✓$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

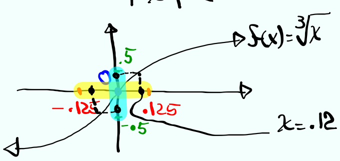
$|\sqrt[3]{x} - 0| < \varepsilon$ $\quad$ " $\quad |x - 0| < \delta$

$|\sqrt[3]{x}| < \varepsilon$ $\quad$ " $\quad |x| < \delta$

Raise both sides by 3rd power

$|\sqrt[3]{x}|^3 < \varepsilon^3$

$|x| < \varepsilon^3$ $\quad \boxed{\text{Pick } \delta = \varepsilon^3}$



Pick $\varepsilon = .5$

then $\delta = .5^3 = \boxed{.125}$

$x = .12$ $\quad \sqrt[3]{.12} \approx .49$

$x = -.1$ $\quad \sqrt[3]{-.1} \approx -.46$

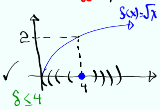
Mar 5-9:35 AM

Prove $\lim_{x \to 4} \sqrt{x} = 2$

1) $f(x) = \sqrt{x}$ $L = 2$ ✓ $a = 4$

2) verify the limit

$\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$ ✓



3) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|\sqrt{x} - 2| < \varepsilon$ " $|x - 4| < \delta$

$\left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| < \varepsilon$ " $|x-4| < \delta$

$\left|\frac{x-4}{\sqrt{x}+2}\right| < \varepsilon$ " $|x-4| < \delta$

keep

$\frac{1 \, |x-4|}{|\sqrt{x}+2|} < \varepsilon$

Bound $\frac{1}{|\sqrt{x}+2|}$

$\boxed{\text{If } \delta \le 4}$

$|x-4| < 4$

$-4 < x-4 < 4$

Add 4

$0 < x < 8$

$\sqrt{0} < \sqrt{x} < \sqrt{8}$

$0 < \sqrt{x} < \sqrt{8} < \sqrt{9}$

$0 < \sqrt{x} < \sqrt{9}$

$0 < \sqrt{x} < 3$

Add 2

$2 < \sqrt{x} + 2 < 5$

Reciprocal

$\frac{1}{2} > \frac{1}{\sqrt{x}+2} > \frac{1}{5}$

$\frac{1}{5} < \frac{1}{\sqrt{x}+2} < \frac{1}{2}$

$\frac{1}{|\sqrt{x}+2|} < \frac{1}{2}$

$\frac{1}{|\sqrt{x}+2|} |x-4| < \frac{1}{2} |x-4| < \varepsilon$

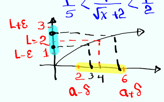
$|x-4| < \frac{\varepsilon}{1/2}$

$|x-4| < 2\varepsilon$

Pick

$\delta = \min\{4, 2\varepsilon\}$

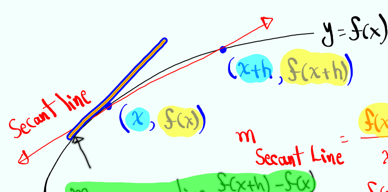
If $\varepsilon = 1$ $\delta = \min\{4, 2\}$
$= 2$



$x = 3 \to \sqrt{3} = 1.73$

$x = 5.999 \to \sqrt{5.999} \approx 2.449$

Mar 3-10:58 AM



$m_{\text{Secant Line}} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$

$m_{\text{tan. line}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

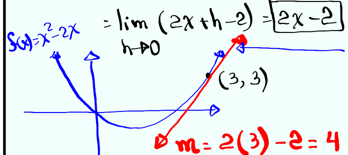
As $h \to 0$, Secant line $\to$ tangent line

Find an expression for the slope of the tangent line to the graph of $f(x) = x^2 - 2x$.

$m_{\text{tan. line}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - x^2 + 2x}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- 2x} - 2h \cancel{- x^2} \cancel{+ 2x}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h - 2)}{\cancel{h}}$

$= \lim_{h \to 0} (2x + h - 2) = \boxed{2x - 2}$



$m = 2(3) - 2 = 4$

$y - y_1 = m(x - x_1)$

$y - 3 = 4(x - 3)$

$\boxed{y = 4x - 9}$

Eqn of Tan. line at $x = 3$.

Mar 5-10:13 AM

Find an expression for the slope of the tangent line to the graph of $f(x)=\frac{1}{x-1}$.

$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

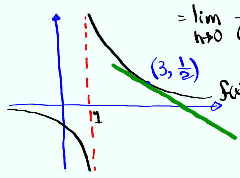
LCD $=(x+h-1)(x-1)$

$$=\lim_{h\to 0}\frac{\frac{1}{x+h-1}-\frac{1}{x-1}}{h}$$

$$=\lim_{h\to 0}\frac{(x-1)\cdot 1-(x+h-1)}{(x+h-1)(x-1)h}$$

$$=\lim_{h\to 0}\frac{\cancel{x}-\cancel{1}-\cancel{x}-h+\cancel{1}}{h(x+h-1)(x-1)}=\lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}(x+h-1)(x-1)}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}=\frac{-1}{(x-1)^2}$$



$$m=\frac{-1}{(3-1)^2}=\frac{-1}{4}$$

$$y-y_1=m(x-x_1)$$

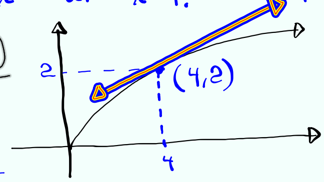
$$y-\frac{1}{2}=\frac{-1}{4}(x-3) \to y=\frac{-1}{4}x+\frac{3}{4}+\frac{1}{2}$$

$$\boxed{y=\frac{-1}{4}x+\frac{5}{4}}$$

Eqn of tan. line at $x=3$.

Mar 5-10:25 AM

Find equation of the tangent line to the graph of $f(x)=\sqrt{x}$ at $x=4$.



$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{(\overset{A}{\sqrt{x+h}}-\overset{B}{\sqrt{x}})(\overset{A}{\sqrt{x+h}}+\overset{B}{\sqrt{x}})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+\cancel{h}-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}=\boxed{\frac{1}{2\sqrt{x}}}$$

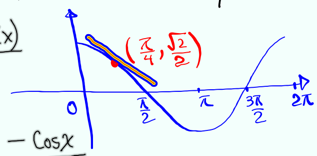
at $x=4 \Rightarrow m=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

$$y-y_1=m(x-x_1)$$

$$y-2=\frac{1}{4}(x-4) \Rightarrow \boxed{y=\frac{1}{4}x+1}$$

Mar 5-10:36 AM

Find slope of the tangent line to the graph of $f(x)=\cos x$ at $x=\frac{\pi}{4}$.



$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos x\cos h-\cos x}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=\lim_{h\to 0}\frac{\cos x(\cos h-1)}{h}-\lim_{h\to 0}\frac{\sin x\sin h}{h}$$

$$=\cos x\underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{\to 0}-\sin x\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{\to 1}$$

$$=\cos x\cdot 0-\sin x\cdot 1=-\sin x$$

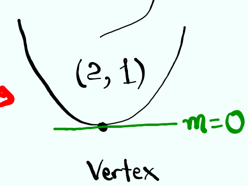
at $x=\frac{\pi}{4}$ $\quad m=-\sin\frac{\pi}{4}$

$$\boxed{m=-\frac{\sqrt{2}}{2}}$$

Mar 5-10:47 AM

Find the point on the graph of $f(x)=x^2-4x+5$ where we have horizontal tan. line.

$f(2)=2^2-4(2)+5$



$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2-4(x+h)+5-x^2+4x-5}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{-4x}-4h\cancel{+5}\cancel{-x^2}\cancel{+4x}\cancel{-5}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h-4)}{\cancel{h}}=\lim_{h\to 0}(2x+h-4)=2x-4$$

we want $m=0$

$2x-4=0$

$\boxed{x=2}$

Mar 5-10:58 AM

Evaluate $\lim_{x \to 4} \frac{\sin(x+4)}{x^2+6x+8} = \frac{\sin(-4+4)}{(-4)^2+6(-4)+8} = \frac{\sin 0}{0} = \frac{0}{0}$

I.F.

$$= \lim_{x \to -4} \frac{\sin(x+4)}{(x+4)(x+2)} = \lim_{x \to -4} \left[\frac{\sin(x+4)}{x+4} \cdot \frac{1}{x+2} \right]$$

$$= \underbrace{\lim_{x \to -4} \frac{\sin(x+4)}{x+4}}_{1} \cdot \lim_{x \to -4} \frac{1}{x+2}$$

$$= 1 \cdot \frac{1}{-4+2} = \boxed{-\frac{1}{2}}$$

Mar 5-11:07 AM

$$\begin{cases} 3 \lim_{x \to 4} f(x) + 2 \lim_{x \to 4} g(x) = 1 \\ \lim_{x \to 4} f(x) - \lim_{x \to 4} g(x) = 2 \end{cases}$$

2

Find $\lim_{x \to 4} f(x)$

$5 \lim_{x \to 4} f(x) = 5$

$$\boxed{\lim_{x \to 4} f(x) = 1}$$

Mar 5-11:12 AM

$$\lim_{x \to 5} f(x) = 4$$

Find $\lim_{x \to 5} \left[x^2 f(x) \right]$

$$= \lim_{x \to 5} x^2 \cdot \boxed{\lim_{x \to 5} f(x)}$$

$$= 5^2 \cdot 4$$

$$= 25 \cdot 4 = \boxed{100}$$

Mar 5-11:15 AM